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Symmetric Random Walks on Three Half-Cubes

Ahmad Barhoumi*, Chung Ching Cheung†, Michael R. Pilla‡ and Jyotirmoy Sarkar§

Abstract

We study random walks on the vertices of three non-isomorphic half-cubes obtained from a cube by a plane cut through its center. Starting from a particular vertex (called the origin), at each step a particle moves, independently of all previous moves, to one of the vertices adjacent to the current vertex with equal probability. We find the means and the standard deviations of the number of steps needed to: (1) return to origin, (2) visit all vertices, and (3) return to origin after visiting all vertices. We also find (4) the probability distribution of the last vertex visited.

Keywords: Absorbing State, Cover Time, Eventual Transition, One-step Transition, Recursive Relation, Transient State.

MSC Codes: 05C81, 97K60

1 Introduction

The cube is a familiar three-dimensional geometric object. A hollow cube is used as a box to store toys, pens, cookies, chocolates, etc. A solid cube is used as a paper weight, or a display case for dates, photographs, etc. The six faces of a

*Department of Mathematics, University of Michigan, Ann Arbor, MI, USA. Email: barhoumi@umich.edu, ORCID: 0000-0001-9034-9326

†Amazon, Austin, TX, USA. Email: cheunt@amazon.com, ORCID: 0000-0002-9223-2452

‡Mathematics, Indiana University, Bloomington, IN, USA. Email: mpilla@iu.edu, ORCID: 0000-0002-3656-6933

§Mathematical Sciences, Indiana University-Purdue University Indianapolis, IN, USA. Email: jsarkar@iupui.edu, ORCID: 0000-0001-5002-5845

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small cube are labelled with numbers 1 through 6 (or with as many dots) and used as a die in parlor games of chance. One of us has studied a random walk (RW) on the vertices of a cube in [9], which also documents a rich literature on the study of RWs on the vertices of graphs, such as [2], [4], [10], [5] and [7]. To develop notation, intuition and techniques, we urge the readers to read [8], which studies symmetric RWs on the vertices of a tetrahedron and an octahedron.

Here we are interested in studying RWs on other simple objects derived from a cube. There are many such objects; but we focus on half-cubes obtained by slicing the cube with a single plane cut passing through its center. The motivation came from a tea party for which two members had signed up to bring cheese cubes; but one of them called in sick. The hostess decided to cut in half all the cheese cubes the other member brought in. Soon she realized that a plane cut through the center makes the halves not only equal in volume, but also equal in all respects! Moreover, as she tilted the cut at various angles, she discovered many different shapes emerging. While there were not enough cheese cubes to go around, there were plenty of cheese hemi-cubes for all members to enjoy. Incidentally, the hemi-cubes served as a great ice breaker at the party.

We shall prove that there are exactly three distinct hemi-cubes whose graph theoretic representations are non-isomorphic to one another and distinct from the graph of the cube itself. It is worth noting that the plane cut generates a new face, some new edges and maybe (or maybe not) some new vertices onto each half-cube. We shall study RWs on the vertices of these three half-cubes.

Why do we study random walks? Other than being intellectually stimulating, a study of random walks prepares one to deal with a sequence of random variables that are not completely independent (because the distribution of each successive outcome depends on the current outcome). Such studies are necessary to equip a researcher to to be able to model complex phenomena such as weather forecasts, stock-market behavior, social network interactions, etc. For a fascinating application of random walks in proving that virtually all polynomials are prime, see [1], or simply read a popular article [3] about that paper. From this article we extract the following quotation, which sufficed to motivate us to conduct this research.

> "If you want to study something and you can't prove a lot of things,
> it's good to start with something simple." —Lior Bary-Soroker

Let us recall the notion of a random walk on a graph, introduce some notation, raise some common questions, and explain the general strategy of answering them.

In general, a RW is a sequence of conditionally independent outcomes determined according to a distribution specific to the current outcome. Specializing to RWs on the vertices of the three hemi-cubes, we imagine that an insect walks along the edges of the hemi-cube. It starts from a particular vertex (which we call the origin). Then at each step the insect moves to any one of the several vertices adjacent (directly connected by an edge) to the current vertex, including the vertex it came from, with equal probability. Thus, we only consider symmetric RWs. We assume that each move, though it depends on the current position, is independent of all previous moves. The insect keeps on moving from one vertex to another until a particular intermediate event happens.

Let $X(t)$ denote the location of the RW at time t. Without loss of generality, the RW starts at Vertex 0; that is, $X(0) \equiv 0$, and Vertex 0 is called the origin. Then at time epochs $t = 1, 2, 3, \ldots$, the RW moves from the current vertex to the next. We answer three questions: What is the probability distribution of the time until the RW

Q1) returns to origin?

Q2) visits all vertices (at least once)?

Q3) returns to origin after visiting all vertices?

The epochs when these three events occur are respectively called the return time T_R, the cover time $\bar{T}$, and the additional time to return after visiting all vertices ${}_LT_R$, where $L = X(\bar{T})$ denotes the vertex that was visited the last among all vertices. Note that $X(T_R) = 0$ and $X({}_LT_R) = 0$; but $L = X(\bar{T}) \neq 0$, implying that $T_R \neq \bar{T}$. Moreover, in order to answer Questions Q2 and Q3, we must also ask:

Q4) What is the distribution of L?

Let us introduce some more terminologies and notation: The *distance* between two vertices i and j is the number of edges on the *shortest path* from Vertex i to Vertex j and vice versa. Let ${}_iT_j$ denote the time (or the number of steps) taken by the RW to go from Vertex i to Vertex j. Then, without loss of generality (by renumbering the vertices, other than the origin, if necessary), we have $T_R = {}_0T_0 = 1 + {}_1T_0$, and ${}_LT_R = {}_LT_0$. Let us denote the mean, the mean square and the variance of ${}_iT_j$, respectively by ${}_ie_j = E[{}_iT_j]$, ${}_is_j = E[{}_iT_j^2]$, and ${}_iv_j = V({}_iT_j) = {}_is_j - {}_ie_j{}^2$; where E denotes mathematical expectation and V denotes variance of a random variable. The positive square root of the variance is called the standard deviation (SD).

To answer the four questions Q1–Q4 posed above, we draw appropriate stochastic transition diagrams in which each node represents a certain combination of three things: the origin, the current vertex and the set of vertices already visited. In each diagram, we describe the transitions between nodes, allowing suitable renumbering of the vertices. A node in which the event of interest occurs is called an *absorbing state*. For time-random variables $T_R, \bar{T}, {}_LT_R$, we report the mean and the SD together with simulated distributions; and for L, we express the exact probability distribution.

Interested readers may read [8] and [7] to gain more insight into the above-mentioned strategy for studying symmetric RWs on the three hemi-cubes. In summary, the answers for a tetrahedron are rather straight-forward because a tetrahedron is equivalent to a complete graph K_4 on four vertices. But the answers for an octahedron are relatively more intricate because its graph is incomplete—three pairs of diametrically opposite vertices are not connected directly by an edge. Also, as in [7] and [8], we shall answer the four questions in the order Q1, Q4, Q2, Q3, for the three hemi-cubes in Sections 3–5. Section 6 will concludes the paper with a summary and directions of future research. Let us first establish in Section 2 that indeed there are three topologically distinct half-cubes.

2 Three Topologically Distinct Hemi-Cubes

A single plane cut through the center of a cube splits it into two identical halves. Among all such hemi-cubes, there are exactly three whose graph theoretic representations are non-isomorphic to one another and distinct from the graph of a cube. We name them rectangular-, hexagonal- and rhombic hemi-cubes, according to the shape of the new face generated by the plane cut. Of course, the cut also creates some new edges; and it may (or may not) create some new vertices.

Theorem 1 *Each of the two identical half cubes obtained by passing a plane through the center of a cube has a graph theoretic representation that is isomorphic either to the entire cube or to one of the three hemi-cubes: rectangular, hexagonal and rhombic; and these four objects are non-isomorphic to one another.*

Proof. Let us impose a Cartesian coordinate system to analyze the cube. Let it be centered at the origin and have sides parallel to one of the axes and of length 2. Consider a cut made by a plane P passing through the origin. Because the two halves are identical, P must contain an even number of the vertices of the cube: To be precise, such a plane is described by an equation of the form $ax+by+cz=0,\ a,b,c\in\mathbb{R}$. In particular, $(x,y,z)\in P \Leftrightarrow (-x,-y,-z)\in P$. The vertices of the cube are precisely all the points with coordinates $x,y,z\in\{\pm1\}$. Hence, P contains an even number of vertices. Furthermore, since at most 4 vertices of a cube can be coplanar, it suffices to let P contain 0, 2, or 4 vertices of the original cube.

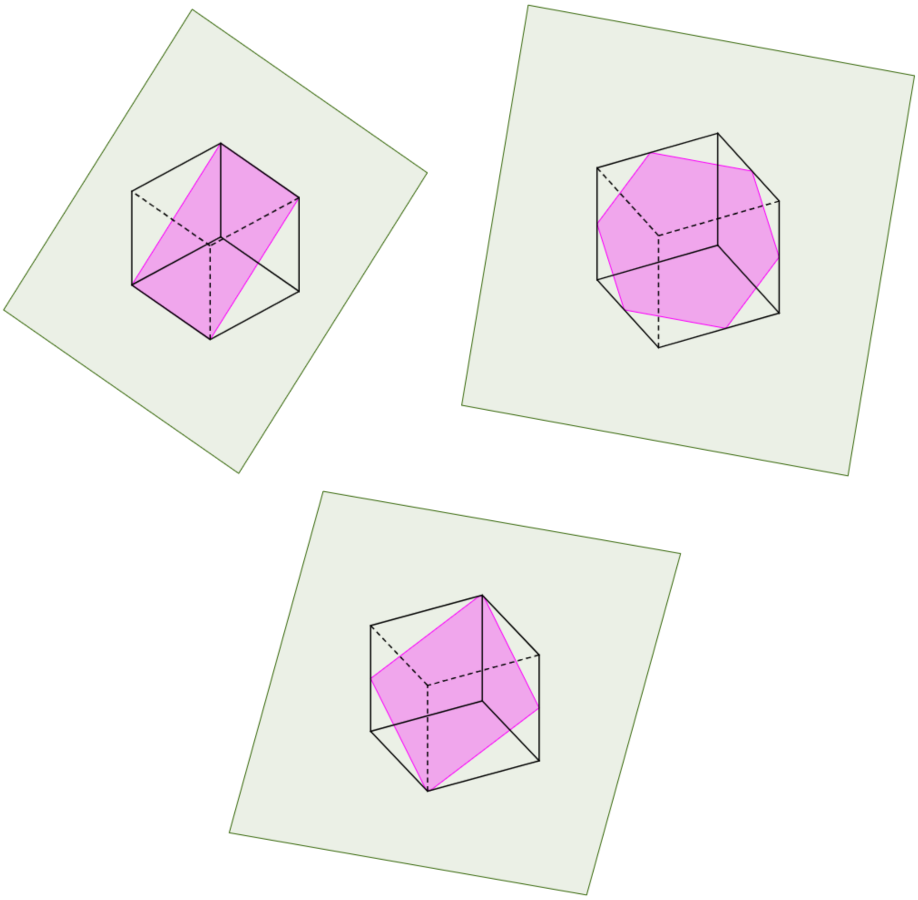

Figure 1: A plane passing through the center of a cube can slice it into one of three topologically distinct hemi-cubes—rectangular-, hexagonal- and rhombic—not isomorphic to one another or to the entire cube.

Suppose that P contains 4 vertices. Since P must also contain the origin, it is unique up to rotation, and such a plane cut P gives rise to the rectangular hemi-cube.

Next, suppose that P contains 2 vertices. These two vertices must be diametrically opposite. Without loss of generality (by suitable rotation), let them be $(1,1,1)$ and $(-1,-1,-1)$. Then P contains exactly one point of the edge $\{(1,-1,z) : z \in (-1,1)\}$, and can be parametrized by this point. Any such plane cut P generates the rhombic hemi-cube, up to isomorphism. In the limit as $z \to 1$ (or as $z \to -1$) the rhombic hemi-cube approaches the rectangular hemi-cube.

Finally, suppose that P does not contain any vertex of the original cube. Then by a similar reasoning as above, it must intersect the cube at either 4 or 6 edges. If P intersects 4 edges, then those 4 edges must be parallel to one another; and the resulting hemi-cube is isomorphic to the original cube. If P intersects 6 edges, then those 6 edges form a connected graph (which may be obtained by removing from the cube the three edges incident at any one vertex and the three edges incident at the diametrically opposite vertex). Such a plane cut P results in the hexagonal hemi-cube, up to isomorphism. ■

3 RW on a Rectangular Hemi-Cube

As far as the symmetric RW is concerned, the simplest among the three half-cubes shown in Figure 1 is the rectangular half-cube. We represent it as a planar graph in Figure 2, labeling the vertices arbitrarily with labels $0, 1, \ldots, 5$.

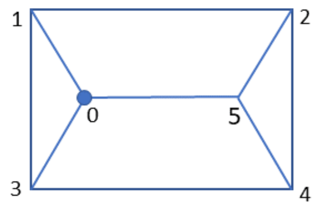

Figure 2: A planar representation of the rectangular hemi-cube obtained by a plane cut containing any two parallel diameters of the cube. Without loss of generality, assume that the RW starts at Vertex 0.

The study of RW on a rectangular hemi-cube is relatively simpler compared to those on other hemi-cubes because it has the fewest number of vertices (only six) and all six vertices are exactly alike in the sense that if any two labels are interchanged, then the remaining four vertices can be relabeled to reconstruct the original labeling. In view of this symmetry, it suffices to study the case when the RW starts from Vertex 0 (origin). We study the RW on the vertices of a rectangular hemi-cube in full detail, answering all four questions, and equipping the readers to imitate the techniques to study the RW on the vertices of the other two hemi-cubes.

Incidentally, the rectangular hemi-cube is also a triangular prism, which is not a distance-regular graph, but it is a vertex-transitive graph. van Slijpe [11] uses it as the simplest example to illustrate that all vertex-transitive graphs have symmetric expected transition time (and squared transition time) between any two vertices i and j; that is, $E({}_iT_j) = E({}_jT_i)$ and $E({}_iT_j^2) = E({}_jT_i^2)$.

3.1 Return to origin

Starting from Vertex 0, in the next epoch, the RW goes to Vertex 1, 3 or 5 with equal probability 1/3. Hence, $E(T_R) = 1 + \frac{1}{3}E({}_1T_0) + \frac{1}{3}E({}_3T_0) + \frac{1}{3}E({}_5T_0)$. By symmetry, we have ${}_1T_0 = {}_3T_0$; consequently, $E(T_R) = 1 + \frac{2}{3}E({}_1T_0) + \frac{1}{3}E({}_5T_0)$. To evaluate $E(T_R)$, we shall express each of $E({}_1T_0)$ and $E({}_5T_0)$ as a function of $E({}_2T_0)$. Thereafter, we shall express $E({}_2T_0)$ as a function of $E({}_1T_0)$ and $E({}_5T_0)$, and hence of itself. Then we shall evaluate $E({}_2T_0)$ and $E(T_R)$.

If the RW moves to Vertex 1, then in the second epoch, it can move to Vertex 0, 2, or 3 with probability 1/3, making $E({}_1T_0) = \frac{1}{3}(1) + \frac{1}{3}E({}_2T_0 + 1) + \frac{1}{3}E({}_3T_0 + 1)$. Using $E({}_3T_0) = E({}_1T_0)$, we have $E({}_1T_0) = \frac{3}{2} + \frac{1}{2}E({}_2T_0)$. On the other hand, if the RW moves to Vertex 5 in the first epoch, then it can move to Vertex 0, 2, or 4 with probability 1/3, yielding $E({}_5T_0) = 1 + \frac{2}{3}E({}_2T_0)$, since $E({}_4T_0) = E({}_2T_0)$. Finally, the RW from Vertex 2 can move to Vertex 1, 4, or 5 with probability 1/3 each, implying that $E({}_2T_0) = 1 + \frac{1}{3}E({}_1T_0) + \frac{1}{3}E({}_4T_0) + \frac{1}{3}E({}_5T_0) = \frac{3}{2} + \frac{1}{2}E({}_1T_0) + \frac{1}{2}E({}_5T_0)$, where we have used $E({}_4T_0) = E({}_2T_0)$.

Next, substituting the just derived expressions for $E({}_1T_0)$ and $E({}_5T_0)$ in terms of $E({}_2T_0)$, we obtain $E({}_2T_0) = \frac{33}{5}$, and $E(T_R) = \frac{7}{3} + \frac{5}{9}E({}_2T_0) = 6$.

In fact, the expected transition time from any other vertex to Vertex 0 can be obtained by solving the above equations:

$$E(T_R) = 6;\ \ E({}_1T_0) = E({}_3T_0) = \frac{24}{5};\ \ E({}_5T_0) = \frac{27}{5};\ \ E({}_2T_0) = E({}_4T_0) = \frac{33}{5}.$$

To evaluate the standard deviation of T_R, we first obtain the expected value of the squares of T_R, ${}_1T_0$, ${}_2T_0$, and ${}_5T_0$. Following the same argument as above, and substituting the expected transition times, we have the following system of equations:

$$\begin{aligned}
E(T_R^2) &= 11 + \frac{2}{3}E({}_1T_0{}^2) + \frac{1}{3}E({}_5T_0{}^2) \\
E({}_1T_0^2) &= \frac{129}{10} + \frac{1}{2}E({}_2T_0^2) \\
E({}_2T_0^2) &= \frac{183}{10} + \frac{1}{2}E({}_1T_0^2) + \frac{1}{2}E({}_5T_0^2) \\
E({}_5T_0^2) &= \frac{49}{5} + \frac{2}{3}E({}_2T_0^2)
\end{aligned}$$

Solving the system gives us $E(T_R^2) = 503/5$ and

$$E({}_1T_0{}^2) = E({}_3T_0{}^2) = \frac{4143}{50};\ \ E({}_5T_0{}^2) = \frac{2577}{25};\ \ E({}_2T_0{}^2) = E({}_4T_0{}^2) = \frac{3498}{25}.$$

So, T_R has standard deviation $SD(T_R) = \sqrt{503/5 - 6^2} = 8.0374$. Similarly,

$$SD({}_1T_0) = SD({}_3T_0) = 7.7343;\ \ SD({}_5T_0) = 8.5977;\ \ SD({}_2T_0) = SD({}_4T_0) = 9.8163.$$

3.2 The vertex visited the last and the cover time

In this subsection, we study the cover time $\bar{T}$ of a rectangular hemi-cube and the last vertex visited $L = X(\bar{T})$. The reader may imagine that there is a cookie at each vertex. A cookie monster travels with the RW. It eats the cookie when it visits a vertex for the first time. Then the cover time $\bar{T}$ is the number of steps needed to eat all the cookies, and $L = X(\bar{T})$ is the vertex corresponding to the last cookie being eaten.

If the RW starts at Vertex 0, by symmetry, L is equally likely to be either Vertex 1 or 3 (which are adjacent to the origin), and L is also equally likely to be either Vertex 2 or 4 (which are 2 steps away from the origin). Hence, let us define the probability for visiting the last vertices as $\alpha = P(L = 1) = P(L = 3)$, $\beta = P(L = 2) = P(L = 4)$, and $\gamma = P(L = 5)$.

As an aid to evaluating α, β, γ, let us define three pairs of RVs and three joint probabilities: For $i = 1, 3, 5$, let L_i denote the last vertex visited and SL_i denote the second last vertex visited when the RW starts at Vertex i. Let

$$\omega_{10}^{(5)} = P(L_1 = 0, SL_1 = 5),\ \ \omega_{30}^{(1)} = P(L_3 = 0, SL_3 = 1),\ \ \omega_{50}^{(1)} = P(L_5 = 0, SL_5 = 1).$$

Lemma 2 *Using* $\omega_{10}^{(5)},\ \omega_{30}^{(1)},\ \omega_{50}^{(1)}$, *one can calculate* α, β, γ *as follows:*

$$1 = 2\alpha + 2\beta + \gamma \tag{1}$$

$$\gamma = \frac{2}{3}\left(\beta + \omega_{10}^{(5)}\right) \tag{2}$$

$$\alpha = \frac{1}{3}\left(\alpha + \omega_{30}^{(1)}\right) + \frac{1}{3}\left(\beta + \omega_{50}^{(1)}\right) \tag{3}$$

Proof. From the definitions of α, β, γ and the law of total probability, follows (1).

To justify (3), we show that $\{L = 5\}$ is a disjoint union $\{L_1 = 5\} \cup \{L_1 = 0, SL_1 = 5\}$. If $\{L = 5\}$, then starting from Vertex 0, the RW must have moved to either Vertex 1 or Vertex 3. By symmetry, without loss of generality, we assume the RW has moved to Vertex 1. Now, let us consider the truncated RW which starts at Vertex 1 and continues until all vertices are visited. Suppose that the last vertex is 5, which happens with probability $P(L_1 = 5) = \beta$, obtained by relabeling the planar graph. Then by augmenting Vertex 0 at the very beginning of this truncated RW, the last vertex for the original RW starting at Vertex 0 is also 5. Hence, $\{L_1 = 5\}$ implies $\{L = 5\}$. On the other hand, for the truncated RW starting at Vertex 1, if the last vertex is Vertex 0 and the second last vertex is 5, then also by augmenting Vertex 0 at the very beginning of this truncated RW, we see that the last vertex of the original RW is 5. Thus, $\{L_1 = 0, SL_1 = 5\}$ implies $\{L = 0\}$. Clearly, these two cases are mutually exclusive and exhaustive. So, (3) holds.

To justify (2), we first assume, without loss of generality, that the last vertex visited by the RW is Vertex 1. In this case, starting from Vertex 0, the RW must move to either Vertex 3 or Vertex 5. Suppose that the RW moves to Vertex 3. Reasoning as in the previous paragraph, we claim that each of $\{L_3 = 1\}$ and $\{L_3 = 0, SL_3 = 1\}$ implies$\{L = 1\}$. However, by relabeling the vertices, we have $\{L_3 = 1\} = \alpha$. This gives the first term on the right hand side (RHS) of (2). On the other hand, starting from Vertex 0, if the RW moves to Vertex 5, we have each of $\{L_5 = 1\}$ and $\{L_5 = 0, SL_5 = 1\}$ implies $\{L = 1\}$. Again, by relabeling the vertices we have $\{L_5 = 1\} = \beta$. This gives the second term on the RHS of (2). Since all these cases are mutually exclusive and exhaustive, (2) holds. ■

It remains to evaluate $\omega_{10}^{(5)}$, $\omega_{30}^{(1)}$, $\omega_{50}^{(1)}$. For each of these cases, we construct a transition diagram, shown in Figures 3–5, where we consider all possible paths of the RW starting from Vertex i (for $i = 1, 3, 5$) until it reaches the second last vertex (SL_i) and then it reaches Vertex 0 the last. A filled circle in each planar graph represents the current vertex, an unfilled circle denotes a vertex visited previously, and a vertex without either type of circles denotes a vertex not yet visited. The eventual transition probabilities are computed by solving a system of linear equations and are shown on each arrow.

Proceeding backwards, we compute the probability of reaching the desired second last (SL) vertex starting from each node (before reaching the last (L) vertex), and document the probability in the figure. For example, in Figure 3(b), if starting from the leftmost (also the rightmost) node in the bottom row the probability of reaching Vertex SL before Vertex L is denoted by p and the same starting from the middle node is denoted by q, then they satisfy $p = (1/3)\ (1) + (1/3)\ p + (1/3)\ q$, solving which we get $p = 2/3$ and $q = 1/3$. Next, we compute the conditional probabilities (shown on the blue arrows) of reaching a node in the bottommost level starting from the nodes in the immediately higher level, using a system of linear equations. For example, the conditional probability of 3/7 between the second last row and the last row is obtained by defining a, b, c as the probabilities of going from the left-, middle- and right-nodes in the left panel of the penultimate row to the bottommost row. They satisfy

$$a = (1 + b + 0)/3; \;\; b = (a + 0 + c)/3; \;\; c = (1 + b + 0)/3$$

Solving this system of equations we get $a = 3/7$. The values $b = 2/7, c = 3/7$ are not needed in the overall calculations and hence are not documented in the figure. Likewise, by defining a, b, c as the probabilities of going from the right-, middle- and left-nodes in the right panel of the penultimate row to the bottommost row, we prove that the eventual transition probability from the rightmost node within the right panel of the penultimate row to the bottommost row is also 3/7. We leave to the reader to justify calculations of the other conditional probabilities.

The probabilities assigned to the topmost nodes in Figures 3–5, respectively yield $\omega_{10}^{(5)} = \frac{13}{168}$; $\omega_{30}^{(1)} = \frac{11}{168}$; and $\omega_{50}^{(1)} = \frac{13}{168}$. Substituting these values in (1)-(3), and solving the system of equations, we have the following result.

Proposition 3 *For a rectangular hemi-cube, starting from Vertex 0 (origin), the Vertex L visited the last has the following distribution:*

$$P\{L=1\} = P\{L=3\} = \alpha = 335/1848 = 0.181277056$$
$$P\{L=2\} = P\{L=4\} = \beta = 29/132 = 0.219696969$$
$$P\{L=5\} = \gamma = 61/308 = 0.198051948$$

To verify our computations, we simulate a RW on a rectangular hemi-cube using R (based on 10^6 iterations) until all vertices are visited. Vertices 1–5 turn out to be the last vertex with frequencies $181318, 220171, 180513, 219723, 198275$, respectively. A chi-square test (with 4 degrees of freedom (df)) of goodness of fit to the probability distribution $(\alpha, \beta, \alpha, \beta, \gamma)$ yielded a test statistic 4.5067, p-value 0.3418. A finer test of equality of probabilities of Vertices 1 and 3 yields a chi-square (with 1 df) test statistic 1.791, p-value 0.1808. Similarly, a test of equality of probabilities of Vertices 2 and 4 yields a chi-square (with 1 df) test statistic 0.45626, p-value 0.4994. Finally, combining the frequencies for Vertices 1 and 3, and Vertices 2 and 4, we estimated (α, β, γ) as $(0.1809, 0.2200, 0.1983)$. A chi-square test of goodness of fit (on 2 df) yielded a test statistic 2.2626, p-value 0.3226. Hence, the simulation study supports our theoretical results.

Next, we study the cover time $\bar{T}$ of a RW on the vertices of a rectangular hemi-cube. In Figure 6, we draw one-step transitions until all vertices are visited at least once, where each arrow represents a conditional probability of 1/3. We label the nodes of the transition diagram by a distinct letter for the number of visited vertices, together with one or two subscripts: The first subscript is the same for all nodes that communicate with one another and the second to distinguish among them. To avoid subscript of a subscript, we may replace the subscripts with arguments within parentheses.

Without loss of generality, the RW leaves Vertex 0, and goes to Vertex 1 with probability $2/3$ and Vertex 5 with probability $1/3$. From Vertex 1, the RW either returns to the origin with probability $1/3$, in which case we interchange the roles of the origin and Vertex 1, and imagine that the RW is still in Vertex 1, or the RW goes to Vertex 3 with probability $1/3$ or Vertex 2 with probability $1/3$. On the other hand, from Vertex 5, the RW either returns to the origin in which case we interchange the role of origin and Vertex 5 and imagine that the RW is still in vertex 5, or the RW goes to Vertex 2 with probability $2/3$. Etc. The process ends in Node F where all six vertices have been visited at least once.

For any node Z, let ${}_ZT_F$ denote the time taken by the RW to go from Node Z to Node F. In our situation, $\bar{T} = {}_AT_F$. Also, let us denote the mean, the mean square and the variance of ${}_ZT_F$ by ${}_Ze_F = E[{}_ZT_F]$, ${}_Zs_F = E[{}_ZT_F{}^2]$, and ${}_Zv_F = V({}_ZT_F) =$

${}_Z s_F - {}_Z e_F{}^2$, respectively. From Figure 6, we can solve the time variables ${}_Z T_F$'s backwards by proceeding from Node F. The probability distributions of ${}_{E(1,1)}T_F$, ${}_{E(1,2)}T_F$, ${}_{E(1,3)}T_F$ are interrelated as follows:

$$
{}_{E(1,1)}T_F = \begin{cases} 1 & \text{with probability } \frac{1}{3} \\ 1 + {}_{E(1,1)}T_F{}' & \text{with probability } \frac{1}{3} \\ 1 + {}_{E(1,2)}T_F & \text{with probability } \frac{1}{3}, \end{cases}
$$

$$
{}_{E(1,2)}T_F = \begin{cases} 1 + {}_{E(1,1)}T_F & \text{with probability } \frac{1}{3} \\ 1 + {}_{E(1,2)}T_F{}' & \text{with probability } \frac{1}{3} \\ 1 + {}_{E(1,3)}T_F & \text{with probability } \frac{1}{3}, \end{cases}
$$

$$
{}_{E(1,3)}T_F = \begin{cases} 1 & \text{with probability } \frac{1}{3} \\ 1 + {}_{E(1,2)}T_F & \text{with probability } \frac{2}{3}. \end{cases}
$$

We will not attempt to get the explicit expressions for the probability distributions of ${}_Z T_F$'s since they are difficult to derive. Instead, we can use Figure 6 to construct the interrelations among the means and the mean squares of ${}_Z T_F$'s. These relations can be solved and we proceed backwards until we get the mean and the mean square of $\bar{T} = {}_A T_F$. A couple of these computations are illustrated as follows.

(1) Taking expectations on the above interrelations, we get

$$
\begin{aligned}
{}_{E(1,1)}e_F &= 1 + (1/3)_{E(1,1)}e_F + (1/3)_{E(1,2)}e_F \\
{}_{E(1,2)}e_F &= 1 + (1/3)_{E(1,1)}e_F + (1/3)_{E(1,2)}e_F + (1/3)_{E(1,3)}e_F \\
{}_{E(1,3)}e_F &= 1 + (2/3)_{E(1,2)}e_F
\end{aligned}
$$

which when solved yield ${}_{E(1,1)}e_F = 24/5$, ${}_{E(1,2)}e_F = 33/5$, ${}_{E(1,3)}e_F = 27/5$. Similarly, first squaring and then taking expectations, we get

$$
\begin{aligned}
{}_{E(1,1)}s_F &= 129/10 + (1/2)_{E(1,2)}s_F \\
{}_{E(1,2)}s_F &= 183/10 + (1/2)_{E(1,1)}s_F + (1/2)_{E(1,3)}s_F \\
{}_{E(1,3)}s_F &= 49/5 + (2/3)_{E(1,2)}s_F
\end{aligned}
$$

which when solved yield ${}_{E(1,1)}s_F = 1212/25$, ${}_{E(1,2)}s_F = 1779/25$, ${}_{E(1,3)}s_F = 1431/25$.

(2) The random time ${}_{D(2)}T_F$ equals $1 + {}_{E(1,3)}T_F$ with probability $1/3$ and $1 + {}_{D(2)}T_F{}'$ with probability $2/3$, where ${}_{D(2)}T_F{}'$ is IID as ${}_{D(2)}T_F$. Hence, by taking expectation, we get

$$
{}_{D(2)}e_F = 1 + (1/3)(27/5) + (2/3)_{D(2)}e_F
$$

which when solved yield ${}_{D(2)}e_F = 42/5$. Similarly, by first squaring and then taking expectation, we have

$$ {}_{D(2)}s_F = 1 + (4/3)_{D(2)}e_F + (2/3)_{D(2)}s_F + (2/3)_{E(1,3)}e_F + (1/3)_{E(1,3)}s_F $$

which when solved yield ${}_{D(2)}s_F = 2616/25$.

Based on the computations shown of Figure 6, we have $E(\bar{T}) = 177/14 = 12.642857$, $var(\bar{T}) = 25881/700 = 36.97286$, and $SD(\bar{T}) = 6.080531$. These exact results are confirmed by our simulation study based on 10^6 iterations yielding estimated expected cover time as 12.6363, with a standard deviation of 6.0779. The truncated simulated cover time distribution is depicted in the next figure.

3.3 Return to origin after visiting all vertices

Recall from Section 3.2 that the last vertex visited is adjacent to the origin with probability $2\alpha + \gamma$, and at a distance two from the origin with probability 2β. In fact, the distribution of the time ${}_LT_R$ to return to the origin after visiting all vertices is given by

$$ {}_LT_R = \begin{cases} {}_1T_0 & \text{with probability } 2\alpha \\ {}_2T_0 & \text{with probability } 2\beta \\ {}_5T_0 & \text{with probability } \gamma \end{cases} $$

In Section 3.1, we established $E({}_1T_0) = 24/5$, $E({}_2T_0) = 33/5$, and $E({}_5T_0) = 27/5$. Hence, the expected value $E({}_LT_R) = 2\alpha E({}_1T_0) + 2\beta E({}_2T_0) + \gamma E({}_5T_0) = 5.709740$. Also, in Section 3.1, we obtained $E({}_1T_0{}^2) = 4143/50$, $E({}_2T_0{}^2) = 3498/25$ and $E({}_5T_0{}^2) = 2577/25$. Hence, $E({}_LT_R{}^2) = 2\alpha E({}_1T_0{}^2) + 2\beta E({}_2T_0{}^2) + \gamma E({}_5T_0{}^2) = 111.9364$, whence by subtracting the square of $E({}_LT_R)$, we obtain the variance $V({}_LT_R) = 79.3353$, and standard deviation $SD({}_LT_R) = 8.9070$.

4 RW on a Hexagonal Hemi-Cube

Let us replicate the findings of Section 3 for a RW on a hexagonal hemi-cube. The planar graph (shown in Fig. 8) is regular (of degree three) on ten vertices. However, unlike the rectangular hemi-cube, the vertices are not completely interchangeable. While the hexagonal hemi-cube has six vertices (freshly generated by the plane cut) on the hexagonal face, it has three vertices of the original cube at a distance one from the hexagonal face, and one vertex of the original cube at a distance two from it. Consequently, the study of a RW on hexagonal hemi-cube poses harder challenges than that on the rectangular hemi-cube. Fortunately, we can use the same line of derivations as in section 3; however, occasionally we shall omit some details.

For the sake of brevity, we assume the RW starts at Vertex 0, the center point in the planar graph above. We leave it to the interested reader to study other possible starting vertex—either Vertex 1 or Vertex 4.

4.1 Return to origin

We assume the RW starts at Vertex 0 (the origin); and without loss of generality, it goes to vertex 1 (after renumbering, if necessary). Thus $T_R = 1 + {}_1T_0$. From Vertex 1, the RW returns to the origin with probability $1/3$, or it goes to Vertex 5 with probability $2/3$, since ${}_5T_0 = {}_9T_0$. If the RW moves to Vertex 5, then it returns to Vertex 1 with probability $1/3$, or it moves to Vertex 8 or Vertex 9 with probability $2/3$. But since ${}_8T_0 = {}_9T_0 = {}_5T_0$, in this case, ${}_5T_0 = 1 + {}_5T_0'$ where ${}_5T_0'$ has the same distribution as ${}_5T_0$. Thus, we have the following systems of equations

$$T_R = 1 + {}_1T_0,$$

$${}_1T_0 = \begin{cases} 1 & \text{with probability } \frac{1}{3} \\ 1 + {}_5T_0 & \text{with probability } \frac{2}{3}, \end{cases}$$

$${}_5T_0 = \begin{cases} 1 + {}_1T_0 & \text{with probability } \frac{1}{3} \\ 1 + {}_5T_0{}' & \text{with probability } \frac{2}{3}. \end{cases}$$

Substituting ${}_1T_0$ on the right hand side of ${}_5T_0$, we rewrite ${}_5T_0$ as a function of random variables ${}_5T_0{}'$ and ${}_5T_0{}''$ having the same distribution as ${}_5T_0$ and all three random variables ${}_5T_0$, ${}_5T_0{}'$ and ${}_5T_0{}''$ being independent. That is,

$${}_5T_0 = \begin{cases} 2 & \text{with probability } \frac{1}{9} \\ 2 + {}_5T_0{}'' & \text{with probability } \frac{2}{9} \\ 1 + {}_5T_0{}' & \text{with probability } \frac{2}{3}. \end{cases}$$

Taking expectations random variables ${}_5T_0$, ${}_5T_0{}'$ and ${}_5T_0{}''$ gives

$$\begin{aligned} E(T_R) &= 1 + E({}_1T_0), \\ E({}_1T_0) &= 1 + \frac{2}{3}E({}_5T_0), \\ E({}_5T_0) &= \frac{4}{3} + \frac{8}{9}E({}_5T_0). \end{aligned}$$

Solving the system of equations backwards, and using symmetry, we obtain

$$\begin{aligned} E({}_4T_0) &= E({}_5T_0) = E({}_6T_0) = E({}_7T_0) = E({}_8T_0) = E({}_9T_0) = 12, \\ E({}_1T_0) &= E({}_2T_0) = E({}_3T_0) = 9, \\ E(T_R) &= 10. \end{aligned}$$

Next, to evaluate the standard deviation of T_R, we take the expected values of squares of T_R, ${}_1T_0$, and ${}_5T_0$ to obtain the following system of equations

$$\begin{aligned} E(T_R^2) &= 1 + 2E({}_1T_0) + E({}_1T_0{}^2) = 19 + E({}_1T_0{}^2) \\ E({}_1T_0^2) &= 1 + \frac{4}{3}E({}_5T_0) + \frac{4}{3}E({}_5T_0^2) = 17 + \frac{2}{3}E({}_5T_0^2) \\ E({}_5T_0^2) &= \left[\frac{4}{9} + \frac{8}{9} + \frac{2}{9}\right] + \left[\frac{8}{9} + \frac{4}{3}\right]E({}_5T_0) + \left[\frac{2}{9} + \frac{2}{3}\right]E({}_5T_0^2) = \frac{86}{3} + \frac{8}{9}E({}_5T_0^2) \end{aligned}$$

Solving the above system, again backwards, we obtain

$$E({}_5T_0{}^2) = 258; \;\; E({}_1T_0{}^2) = 189; \;\; E(T_R^2) = 208.$$

Hence, the standard deviation of T_R is $SD(T_R) = \sqrt{208 - 10^2} = 6\sqrt{3} = 10.3923$.

Remark 1. It is a pleasant surprise that for a rectangular hemi-cube, $E[T_R] = 8$, the number of vertices of a rectangular hemi-cube; and for a hexagonal hemi-cube, $E[T_R] = 10$, the number of vertices of a hexagonal hemi-cube. Indeed, a similar result holds for all regular graphs. The intuition behind this phenomenon is that in the long-run (that is, as $t \to \infty$) the RW forgets its starting vertex, and its chance of being at a particular vertex is proportional to the degree of that vertex. See [5]. In particular, for a regular graph, the RW is equally likely to be at any vertex. This in turn means that the expected number of steps between successive visits to any particular vertex is the reciprocal of the probability that the RW is at that vertex, which is the same as the number of vertices.

4.2 The vertex visited the last and the cover time

We next study the cover time $\bar{T}$ and the last vertex visited $L = X(\bar{T})$ on a hexagonal hemi-cube. By symmetry L is equally likely to be vertices 1, 2, or 3 (which are adjacent to the origin) and L is equally likely to be vertices $4, 5, 6, 7, 8,$ or 9 (which are two steps away from the origin). Let us define the probability that a vertex will be the last one visited as

$$\alpha = P(L = 1) = P(L = 2) = P(L = 3);$$

and

$$\beta = P(L = 4) = P(L = 5) = P(L = 6) = P(L = 7) = P(L = 8) = P(L = 9).$$

To evaluate α and β, we consider a RW that starts at a designated vertex i and continues until all vertices are visited resulting in a last vertex j and second last vertex k. As in Section 3, we write $\omega_{ij} = P(L_i = j)$ and $\omega_{ij}^{(k)} = P(L_i = j, SL_i = k)$.

As done in Section 3, we shall compute $\omega_{ij}^{(k)}$ using transition diagrams that keep track of all configurations of triplets consisting of starting vertex i, set of visited vertices, and the current vertex. Thereafter, we shall calculate α and β using the following lemma.

Lemma 4 *After calculating all* $\omega_{ij}^{(k)}$*'s, one can calculate* α *and* β *by solving the following system of six equations:*

$$
\begin{aligned}
1 &= 3\alpha + 6\beta \\
\alpha &= \frac{2}{3}\left[\omega_{21} + \omega_{20}^{(1)}\right] \\
\omega_{21} &= \frac{1}{3}\left[\alpha + \omega_{02}^{(1)}\right] + \frac{1}{3}\left[\omega_{61} + \omega_{62}^{(1)}\right] + \frac{1}{3}\left[\omega_{71} + \omega_{72}^{(1)}\right] \\
\omega_{61} &= \frac{1}{3}\left[\omega_{71} + \omega_{76}^{(1)}\right] + \frac{1}{3}\left[\omega_{21} + \omega_{26}^{(1)}\right] + \frac{1}{3}\left[\omega_{91} + \omega_{96}^{(1)}\right] \\
\omega_{71} &= \frac{1}{3}\left[\omega_{61} + \omega_{67}^{(1)}\right] + \frac{1}{3}\left[\omega_{21} + \omega_{27}^{(1)}\right] + \frac{1}{3}\left[\omega_{41} + \omega_{47}^{(1)}\right] \\
\omega_{91} &= \frac{1}{3}\left[\omega_{51} + \omega_{59}^{(1)}\right] + \frac{1}{3}\left[\omega_{61} + \omega_{69}^{(1)}\right]
\end{aligned}
$$

These equations are easily derived from the fact that if the RW goes from Vertex i to Vertex h in the first step, then the event $\{L_i = j\}$ becomes the disjoint union of events $\{L_h = j\}$ and $\{L_h = i, SL_h = j\}$, implying that $\omega_{ij} = \omega_{hj} + \omega_{hi}^{(j)}$. Additionally, since the planar graph exhibits a reflection symmetry about the line joining vertices 0 and 1, the following probability equalities hold: $\omega_{21} = \omega_{31}$, $\omega_{41} = \omega_{71}$, $\omega_{51} = \omega_{91}$, $\omega_{61} = \omega_{81}$.

It remains to evaluate the twelve joint probabilities $\omega_{ij}^{(k)} = P(L_i = j, SL_i = k)$ used in Lemma 4. We construct the corresponding twelve transition diagrams and evaluate these joint probabilities in the same manner as was done in Section 3 for a RW on the rectangular hemi-cube. For the sake of brevity, we omit the diagrams from this paper. Interested readers may contact the corresponding author. Below we will simply report the exact values of the joint probabilities.

$$\begin{array}{ll|ll}
\omega_{20}^{(1)} = & \frac{482189651753}{19014656192532} & \omega_{02}^{(1)} = & \frac{147390259}{19986553587} \\
\omega_{62}^{(1)} = & \frac{9078538429}{1482638884272} & \omega_{72}^{(1)} = & \frac{25848990283}{4077256931748} \\
\omega_{76}^{(1)} = & \frac{147450804923}{22065155160048} & \omega_{26}^{(1)} = & \frac{699969775}{105574905072} \\
\omega_{96}^{(1)} = & \frac{81693040}{1441036779} & \omega_{67}^{(1)} = & \frac{316175689}{54046635552} \\
\omega_{27}^{(1)} = & \frac{886148145955}{159734831373936} & \omega_{47}^{(1)} = & \frac{264701690733}{35496629194208} \\
\omega_{59}^{(1)} = & \frac{94571979868958617}{3939380411344009632} & \omega_{69}^{(1)} = & \frac{529556182610465}{15570673562624544}.
\end{array}$$

Substituting these joint probabilities into the equations in Lemma 4, and then solving the system of equations simultaneously, we obtain

$$\begin{aligned}
\omega_{21} &= \omega_{31} = \frac{124067976665700581}{984845102836002408} \approx 0.1259771474, \\
\omega_{41} &= \omega_{71} = \frac{65990399202175951}{492422551418001204} \approx 0.1340117324, \\
\omega_{51} &= \omega_{91} = \frac{16223819896798627}{179062745970182256} \approx 0.0906041053, \\
\omega_{61} &= \omega_{81} = \frac{242649190694866663}{1969690205672004816} \approx 0.1231915506,
\end{aligned}$$

and more importantly, we obtain the following result.

Proposition 5 *For a hexagonal hemi-cube, starting from Vertex 0, the Vertex L visited the last has the following probabilities:*

$$\begin{aligned}
\alpha &= P\{L=1\} = P\{L=2\} = P\{L=3\}, \\
\beta &= P\{L=4\} = P\{L=5\} = P\{L=6\} = P\{L=7\} = P\{L=8\} = P\{L=9\}.
\end{aligned}$$

where α *and* β *evaluates to*

$$\begin{aligned}
\alpha &= \frac{366107854307753}{3628758669255572} \approx 0.1008906592 \\
\beta &= \frac{1940629641175501}{16692289878576312} \approx 0.1162590427.
\end{aligned}$$

We simulated in R (10^6 iterations of) a RW on a hexagonal hemi-cube until all vertices are visited. The observed frequencies of visiting Vertices 1–9 the last were respectively:

$$100694, 100546, 100719, 116272, 116090, 115761, 116736, 116181, 117001.$$

An overall chi-square test (with 8 df) of goodness of fit has a test statistic 10.929, p-value 0.2057. A finer chi-square test (with 2 df) of equal probabilities for L to be Vertex 1, 2 or 3 had a test statistic 0.17373, p-value 0.9168. Another independent chi-square test (with 5 df) of equal probabilities for L to be one of Vertices 4–9 had a test statistic 8.7793, p-value = 0.1182. Thus, the simulation study corroborates the obvious symmetry that L is equally likely to be Vertices 1, 2, 3; and that L is equally likely to be the last six vertices. Consequently, the aggregate frequencies of the last vertex falling within subsets $\{1,2,3\}$ and $\{4,5,\ldots,9\}$ are 301959 and 69041, respectively. Yet another independent chi-square test of proportion (with 1 df) for $p = 3\alpha = .302672$, yielded a test statistic 1.9684, p-value 0.1606. Thus, the simulation study supports our theoretical results.

Next, we study the cover time $\bar{T}$ of a RW on the vertices of a hexagonal hemi-cube. As in the case of the rectangular hemi-cube, we only keep track of the pattern of visited vertices and the current vertex. We draw one-step transitions until all vertices are visited at least once. Without loss of generality, the RW leaves Vertex 0, and goes to Vertex 1 with probability 1. From Vertex 1, the RW either returns to the origin with probability $\frac{1}{3}$, or it goes to vertex 5, without loss of generality, with probability $\frac{2}{3}$. Etc. Proceeding as in the case of the rectangular hemi-cube, with details omitted, we evaluate

$$E(\bar{T}) = \frac{883611158109394216421}{26248737755545110806} \approx 33.66299615.$$

Likewise, we evaluate the mean squared cover time, $E(\bar{T}^2)$, to be

$$\frac{481263803531309239816840904264209750100780}{348503908936430268404512268000123042781} \approx 1380.94234007.$$

Next, subtracting $E^2(\bar{T})$ and taking square root, the standard deviation is

$$SD(\bar{T}) \approx 15.73991836.$$

A simulation using 10^6 iterations estimates $E(\bar{T})$ as 33.649184, and SD($\bar{T}$) as 15.73503847. The truncated simulated distribution is depicted in the next figure.

4.3 Return to origin after visiting all vertices

In this subsection, we study the time to return to origin after visiting all vertices on a hexagonal hemi-cube. Recall that, for the RW on the hexagonal hemi-cube starting from Vertex 0, the last vertex visited is adjacent to the origin with probability $3\alpha = 1 - 6\beta$ and at a distance two from the origin with probability 6β where $\beta = \frac{1940629641175501}{16692289878576312}$. Moreover, we already established that $E({}_1T_0) = 9$ and $E({}_4T_0) = 12$. Therefore, the expected return time to Vertex 0 after visiting all vertices is given by

$$E({}_LT_R) = 3\alpha E({}_1T_0) + 6\beta E({}_4T_0) = 9(1-6\beta) + 12\cdot 6\beta = 9 + 18\beta \approx 11.09266.$$

Next, using the results from above, we note that

$$E({}_L T_R{}^2) = (1-6\beta)E({}_1T_0{}^2) + 6\beta E({}_5T_0{}^2) = 189 + 414\beta \approx 237.1312.$$

Therefore, we have

$$SD({}_LT_R) = \sqrt{E({}_LT_R{}^2) - E({}_LT_R)^2} = 3\sqrt{12 + 10\beta - 36\beta^2} \approx 10.6810.$$

5 RW on a Rhombic Hemi-Cube

In this section, we study a RW on a rhombic hemi-cube whose planar graph is shown in Fig. 10. In contrast to the other hemi-cubes studied in Sections 3 and 4, this graph, on seven vertices, is not even a regular graph: It has degree four at Vertex 0 and degree three at all other vertices. Consequently, the study of a RW on a rhombic hemi-cube poses much harder challenges than those on the other hemi-cubes. We still follow the same line of derivations as in Section 3; but we omit a lot of details.

Because of reflection symmetry about the line joining vertices 0, 2 and 5, of the planar graph in Fig. 10, we can simultaneously interchange within-pair vertices $(1,3),(4,6)$ without changing the graph. Furthermore, if we simultaneously interchange within-pair vertices $(1,4),(2,5),(3,6)$, we recover the same graph. Therefore, it suffices to consider RWs that begin at Vertex 0, 1 or 2. For brevity, we will study RWs starting at Vertex 0, leaving the other starting vertices to the reader.

5.1 Return to origin

Suppose that the RW begins at Vertex 0 and without loss of generality (up to renumbering) moves to Vertex 1 in the first step. Using the same notation as in the previous cases, we arrive at $T_R = {}_0T_0 = 1 + {}_1T_0$,

$${}_1T_0 = \begin{cases} 1 & \text{w.p. } \frac{1}{3} \\ 1 + {}_2T_0 & \text{w.p. } \frac{1}{3} \\ 1 + {}_4T_0 & \text{w.p. } \frac{1}{3} \end{cases},$$

and

$${}_2T_0 = \begin{cases} 1 + {}_1T_0 & \text{w.p. } \frac{1}{3} \\ 1 + {}_3T_0 & \text{w.p. } \frac{1}{3} \\ 1 + {}_5T_0 & \text{w.p. } \frac{1}{3} \end{cases}.$$

The symmetries mentioned above cause the RVs within subsets $\{{}_1T_0, {}_3T_0, {}_4T_0, {}_6T_0\}$ and $\{{}_2T_0, {}_5T_0\}$ to have the same distribution. Using these distributional equalities and taking expectations in the stochastic interrelations between ${}_1T_0$ and ${}_2T_0$, we obtain two linear equations in two unknowns, solving which we obtain

$$E({}_1T_0) = E({}_3T_0) = E({}_4T_0) = E({}_6T_0) = 4.5,$$
$$E({}_2T_0) = E({}_5T_0) = 6.$$

Thereafter, we have $E(T_R) = E({}_0T_0) = 1 + 4.5 = 5.5$. This result agrees with Remark 1, which says that since the degree is 4 at Vertex 0, and 3 at each of the remaining six vertices for a total of 22, we must have $E(T_R) = 22/4$.

Next, squaring ${}_1T_0$ and ${}_2T_0$, and then taking expectations and simplifying, we obtain $2E({}_1T_0^2) = 24 + E({}_2T_0^2)$ and $2E({}_2T_0^2) = 33 + 2E({}_2T_0^2)$, solving which we get $E({}_2T_0^2) = 57$ and $E({}_1T_0^2) = 40.5$. Hence, we have $SD(T_R) = SD({}_1T_0^2) = \sqrt{40.5 - 4.5^2} = 4.5$.

5.2 The vertex visited the last and the cover time

We now study the cover time $\overline{T}$ and the last vertex visited $L = X(\overline{T})$ for a RW on the vertices of a rhombic hemi-cube.

Consider a RW that begins at vertex 0. Recall the notation $\omega_{ij} = P\{L_i = j\}$ where L_i is the last vertex visited by a RW starting at vertex i for $i = 0, 1, \ldots, 6$. In view of the symmetries of the rhombic hemi-cube, we immediately see that

$$\alpha = \omega_{01} = \omega_{03} = \omega_{04} = \omega_{06} \text{ and } \beta = \omega_{02} = \omega_{05}$$

Likewise, $\omega_{15} = \omega_{35}$, $\omega_{45} = \omega_{65}$. Furthermore, recall the definitions of the joint probabilities $\omega_{ij}^{(k)} = P\{L_i = j, SL_i = k\}$.Using the symmetries, we claim that $\omega_{12}^{(5)} = \omega_{32}^{(5)}$. Then we arrive at the following system of equations:

$$\begin{aligned}
1 &= 4\omega_{01} + 2\omega_{05} \\
\omega_{05} &= \frac{1}{2}\left[\omega_{15} + \omega_{10}^{(5)}\right] + \frac{1}{2}\left[\omega_{45} + \omega_{40}^{(5)}\right] \\
\omega_{15} &= \frac{1}{3}\left[\omega_{05} + \omega_{01}^{(5)}\right] + \frac{1}{3}\left[\omega_{25} + \omega_{21}^{(5)}\right] + \frac{1}{3}\left[\omega_{45} + \omega_{41}^{(5)}\right] \\
\omega_{25} &= \frac{2}{3}\left[\omega_{15} + \omega_{12}^{(5)}\right] \\
\omega_{45} &= \frac{1}{3}\left[\omega_{05} + \omega_{04}^{(5)}\right] + \frac{1}{3}\left[\omega_{15} + \omega_{14}^{(5)}\right]
\end{aligned}$$

As in the previous two cases, we can evaluate the eight joint probabilities $\omega_{ij}^{(k)}$'s using transition diagrams. The diagrams are omitted from this paper, for the sake of brevity. Interested readers may contact the corresponding author. The joint probabilities are

$\omega_{01}^{(5)}$	=	$\frac{3294731}{301862160}$	$\omega_{04}^{(5)}$	=	$\frac{271930027}{4796254320}$
$\omega_{10}^{(5)}$	=	$\frac{4}{1155}$	$\omega_{12}^{(5)}$	=	$\frac{13}{240}$
$\omega_{14}^{(5)}$	=	$\frac{37103603}{550389840}$	$\omega_{21}^{(5)}$	=	$\frac{200002}{11319831}$
$\omega_{40}^{(5)}$	=	$\frac{4}{495}$	$\omega_{41}^{(5)}$	=	$\frac{9499}{511920}$

Thereafter, substituting these joint probabilities in the above set of equations and solving the system of linear equations, we arrive at

$$P(L_4=5)=P(L_6=5)=P(L_1=2)=P(L_3=2)=\frac{79028107}{496164240},$$
$$P(L_2=5)=P(L_5=2)=\frac{1760425958}{11331150831},$$
$$P(L_1=5)=P(L_3=5)=P(L_4=2)=P(L_6=2)=\frac{54049820453}{302164022160},$$

and most importantly

$$\alpha=\omega_{01}=P(L_0=1)=P(L_0=3)=P(L_0=4)=P(L_0=6)=\frac{438611383}{2697893055},$$
$$\beta=\omega_{05}=P(L_0=5)=P(L_0=2)=\frac{943447523}{5395786110}.$$

We simulated in R (10^6 iterations of) a RW on a rhombic hemi-cube starting from Vertex 0 until all vertices are visited. The observed frequencies of visiting Vertices 1–6 the last were respectively:

$$162173, 175006, 163463, 162126, 174516, 162716.$$

An overall chi-square test (with 5 df) of goodness of fit has a test statistic 7.9806, p-value 0.1573. A finer chi-square test (with 3 df) of equal probabilities for L to be Vertex 1, 3, 4 or 6 had a test statistic 7.156, p-value 0.06709. Another independent chi-square test (with 1 df) of equal probabilities for L to be one of Vertices 2 or 5 had a test statistic 0.68694, p-value 0.4072. Thus, the simulation supports the hypotheses that L is equally likely to be Vertices 1, 3, 4, 6; likewise, L is equally likely to be Vertices 2, 5. Consequently, using the aggregate frequencies of the last vertex falling within subsets $\{1,3,4,6\}$ and $\{2,5\}$ being 650478 and 349522, respectively, another independent chi-square test of proportion (with 1 df) for $p=4\alpha=0.6503021$, yielded a test statistic 0.13605, p-value 0.7122. Thus, the simulation study supports our theoretical results.

We leave the deductive study of the cover time $\bar{T}$ for a RW on the vertices of a rhombic hemi-cube to the interested reader. We simply report the simulation results: Based on 10^6 iterations, repeated four times, the estimates of $E(\bar{T})$ (and SD($\bar{T}$)) are: 17.410171 (8.103666), 17.429389 (8.124514), 17.407281 (8.107587), 17.410436 (8.115199). The closeness of the estimates is a testament to the reliability of the measures. The truncated simulated distribution, based on the first simulation, with seed 789, is depicted in the next figure.

5.3 Return to origin after visiting all vertices

Suppose that the RW starts at Vertex 0. Recall that, for the RW on the rhombic hemi-cube starting from Vertex 0, the last vertex visited is adjacent to the origin with probability $4\alpha = 1-2\beta$ and at a distance two from the origin with probability 2β where $\beta = 943447523/5395786110$. Moreover, we already established that $E({}_1T_0) = 9/2$ and $E({}_4T_0) = 6$. Therefore, the expected return time to Vertex 0 after visiting all vertices is given by

$$E({}_LT_R) = 4\alpha E({}_1T_0) + 2\beta E({}_2T_0) = 4.5(1-2\beta) + 6\cdot 2\beta = 4.5 + 3\beta \approx 5.024547.$$

Likewise, $E({}_LT_R^2) = 4\alpha E({}_1T_0^2) + 2\beta E({}_2T_0^2) = 40.5(1-2\beta) + 57\cdot 2\beta = 40.5 + 33\beta \approx 46.27002$. Hence, $\text{SD}({}_LT_R) = \sqrt{46.27002 - 5.024547^2} = 4.584187$.

6 Conclusion and Future Research

We have studied symmetric RWs on the vertices of three topologically non-isomorphic hemi-cubes that are distinct from the cube itself. Specifically, we have studied the mean and the SD of the time until return to origin, the time until all vertices are visited and the additional time until return to origin after visiting all vertices. Alongside, we also studied the exact probability distribution of the last vertex visited.

For the rectangular half-cube, because of symmetry, any vertex can serve as the origin. However, for the hexagonal and the rhombic half-cubes, we have only studied one choice of the origin. For each of these half-cubes, there are two other distinct choices for the origin. We leave the study of RWs starting from these alternative origins to the inquisitive reader. Nonetheless, in Table 1, we document some percentiles, the mean, and the SD of the cover time distribution using simulation based on 10^6 iterations.

We invite interested readers to study some asymmetric RWs on the vertices of hemi-cubes. One form of asymmetry is to consider the edge along which the RW has reached the current vertex as a special edge labeled 0 and the other edges

Table 1: Some percentiles, mean and SD of cover time distributions of RWs on the three half-cubes with different starting vertex

half-cube	start at	$q(.25)$	$q(.50)$	mean	$q(.75)$	$q(.99)$	SD
rectangular	Vertex 0	08	11	12.6363	15	33	6.0779
hexagonal	Vertex 0	22	30	33.6492	41	85	15.7350
	Vertex 1	22	30	33.4523	41	84	15.5882
	Vertex 4	22	30	33.4463	41	84	15.5880
rhombic	Vertex 0	12	16	17.4102	21	44	8.1037
	Vertex 1	11	15	16.7937	21	43	7.9632
	Vertex 2	11	15	16.6903	20	43	7.8500

incident at the current vertex are labeled 1, 2, ... going clockwise from edge 0 when looked at from outside the hemi-cube. Then one can assume that starting from the origin the RW is equally likely to go to any adjacent vertex; but thereafter at each successive step, the next vertex is chosen according to a pre-specified one-step transition probability vector $\mathbf{p}_i = (p_{i,0}, p_{i,1}, p_{i,2}, \ldots)$, where $p_{i,h}$ is the probability of traveling along edge h starting from Vertex i.

We leave open the study of (symmetric or asymmetric) RWs on the faces of each hemi-cube.

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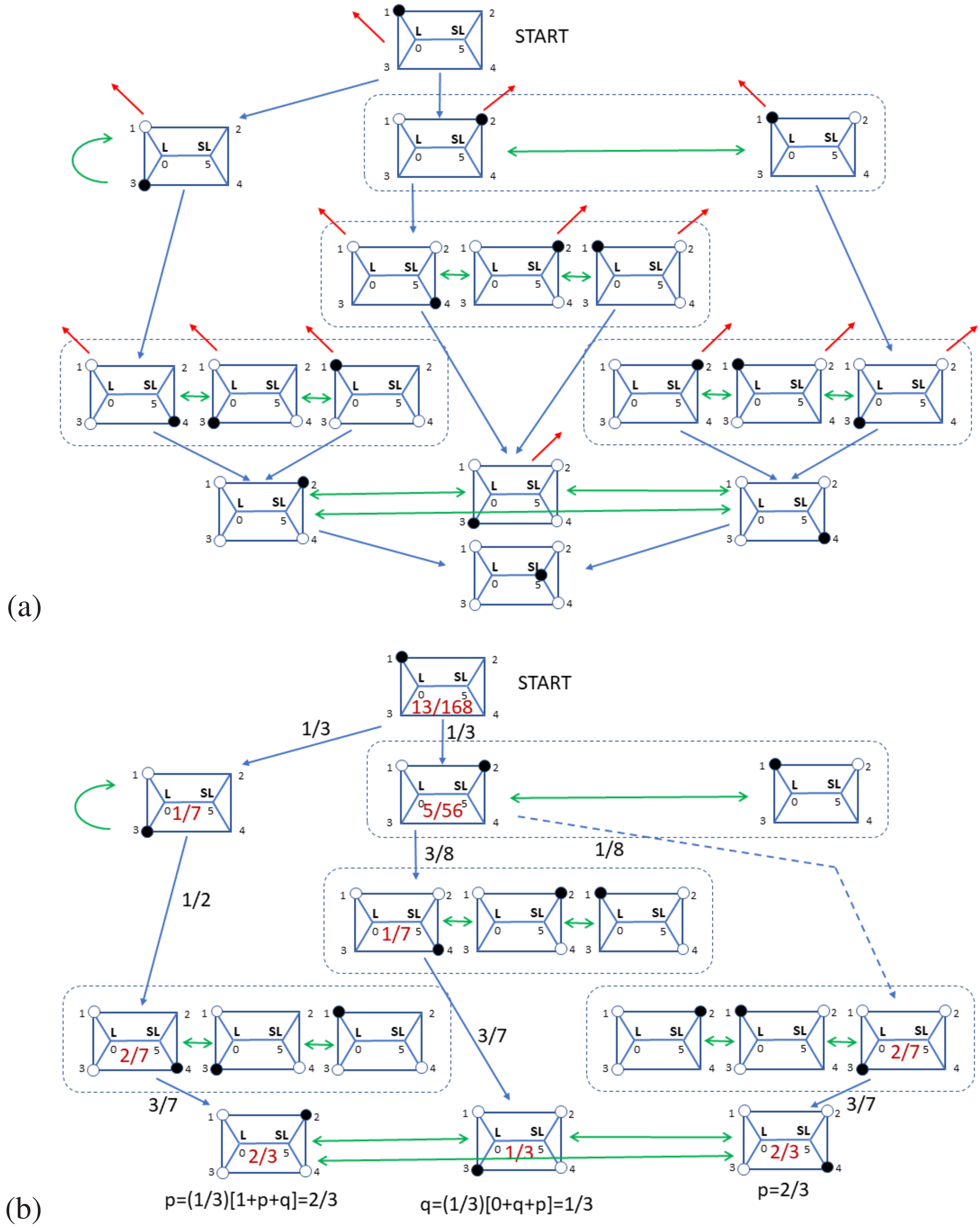

Figure 3: (a) To compute $\omega_{10}^{(5)}$, starting from Vertex 1, we record all possible one-step transitions (with probability 1/3 each). A red arrow indicates a visit to Vertex 5 or Vertex 0. (b) Next, starting from Vertex 1, we record the conditional probabilities of eventual transition to a new vertex, other than Vertices 5 and 0: For nodes in the bottommost row, we compute the conditional probabilities the RW reaches Vertex 5 before Vertex 0 (origin). (A dashed arrow indicates that although there is no direct transition, there is an eventual transition.) Finally, traveling backwards, we multiply along each branch (and add when branches join together) the conditional probabilities to determine $\omega_{10}^{(5)} = 13/168$.

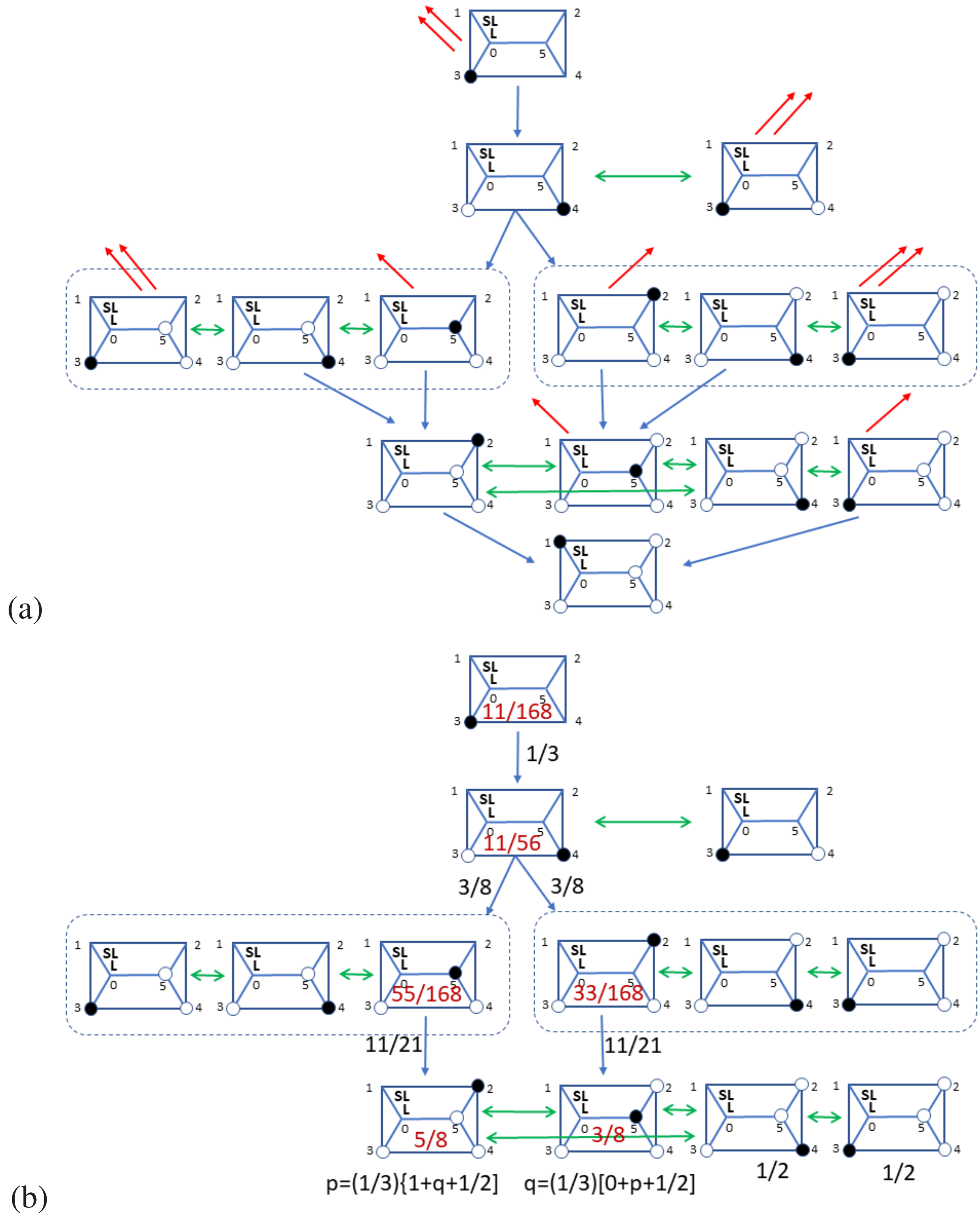

Figure 4: (a) To compute $\omega_{30}^{(1)}$, starting from Vertex 3, we record all possible one-step transitions (with probability 1/3 each). A red arrow indicates a visit to Vertex 1 or Vertex 0. (b) Next, starting from Vertex 3, we record step-by-step the conditional probabilities of eventual transition to a new vertex, other than Vertices 1 and 0. Thereafter, traveling backwards, we compute the conditional probabilities the RW reaches Vertex 1 before Vertex 0 (origin) starting from specific nodes, until we determine $\omega_{30}^{(1)} = 11/168$.

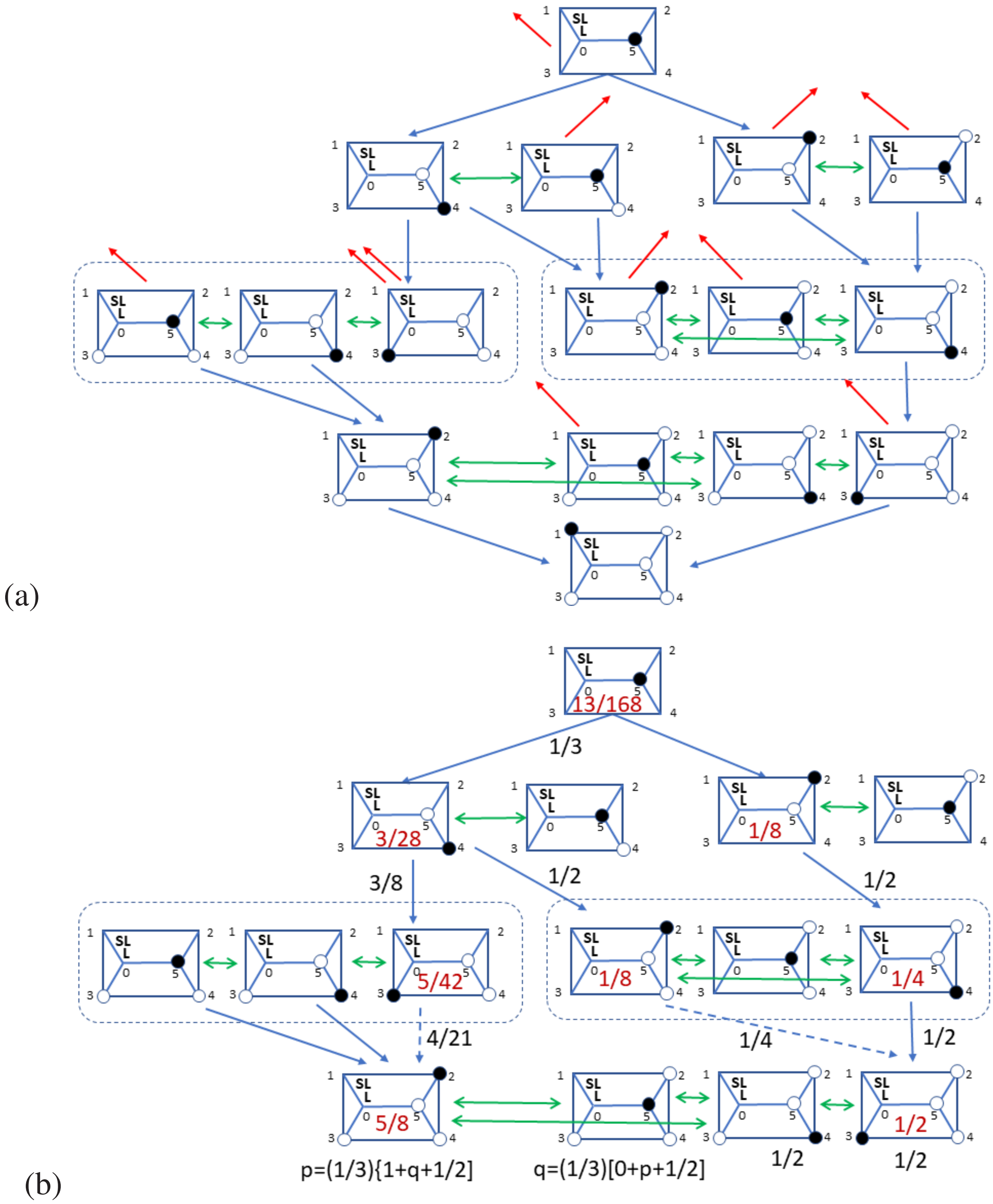

Figure 5: (a) To compute $\omega_{50}^{(1)}$, starting from Vertex 5, we record all possible one-step transitions (with probability 1/3 each). A red arrow indicates a visit to Vertex 1 or Vertex 0. (b) Next, starting from Vertex 5, we record step-by-step the conditional probabilities of eventual transition to a new vertex, other than Vertices 1 and 0. (A dashed arrow indicates that although there is no direct transition, there is an eventual transition.) Thereafter, traveling backwards, we compute the conditional probabilities the RW reaches Vertex 1 before Vertex 0 (origin) starting from specific nodes, until we determine $\omega_{50}^{(1)} = 13/168$.

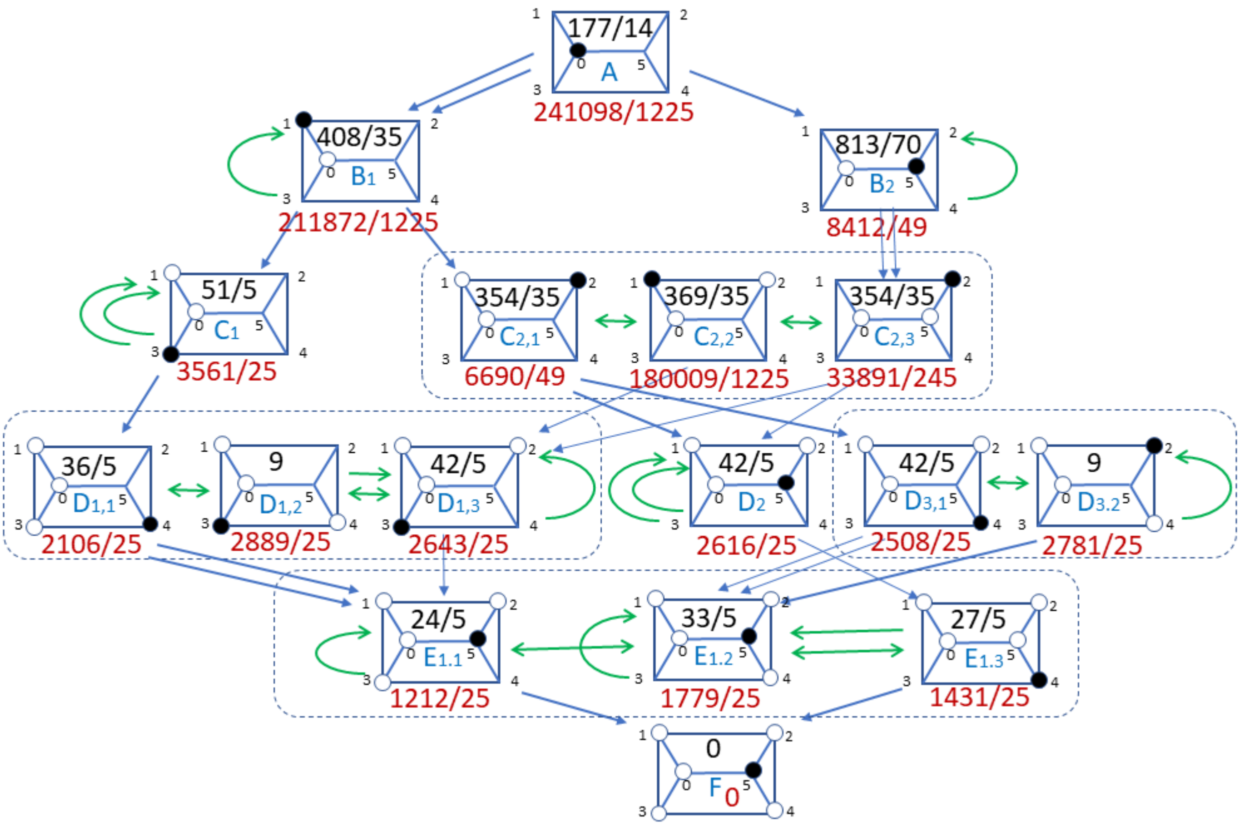

Figure 6: One-step transitions from the start (Node A) until all vertices are visited (Node F). Moving backwards, we calculate from each Node Z, the expected time (black) and the expected squared time (red) to reach Node F.

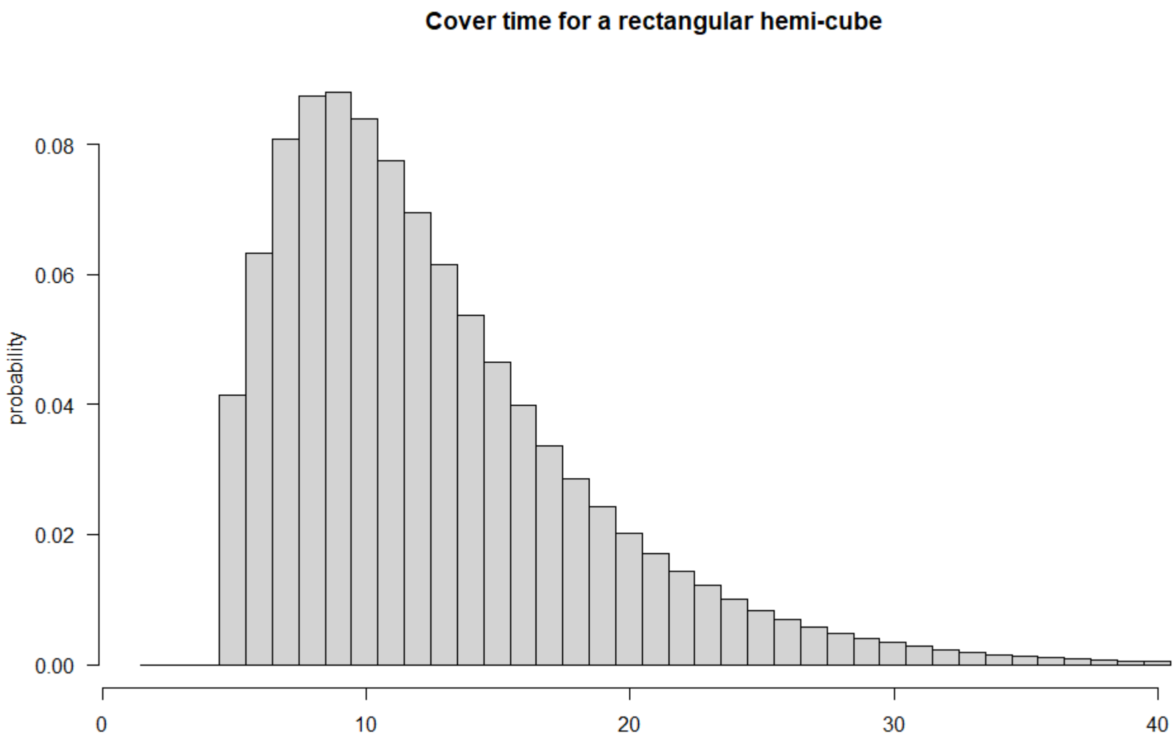

Figure 7: The simulated cover time distribution based on 10^6 iterations, truncated at 40 with $P(\bar{T} > 40) = .002443$. Some estimated quantiles are $q(.25) = 8, q(.50) = 11, q(.75) = 15, q(.99) = 33$.

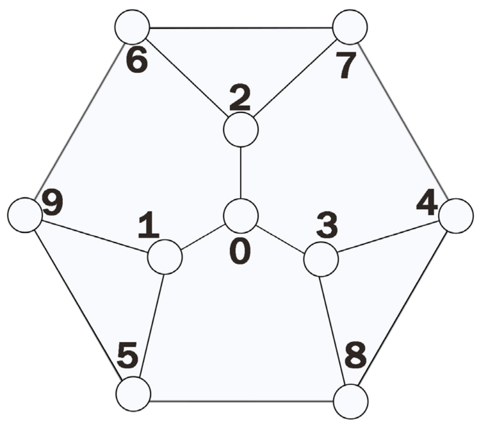

Figure 8: A planar representation of the hexagonal hemi-cube obtained by a plane cut through the center of the cube and orthogonal to a diagonal.

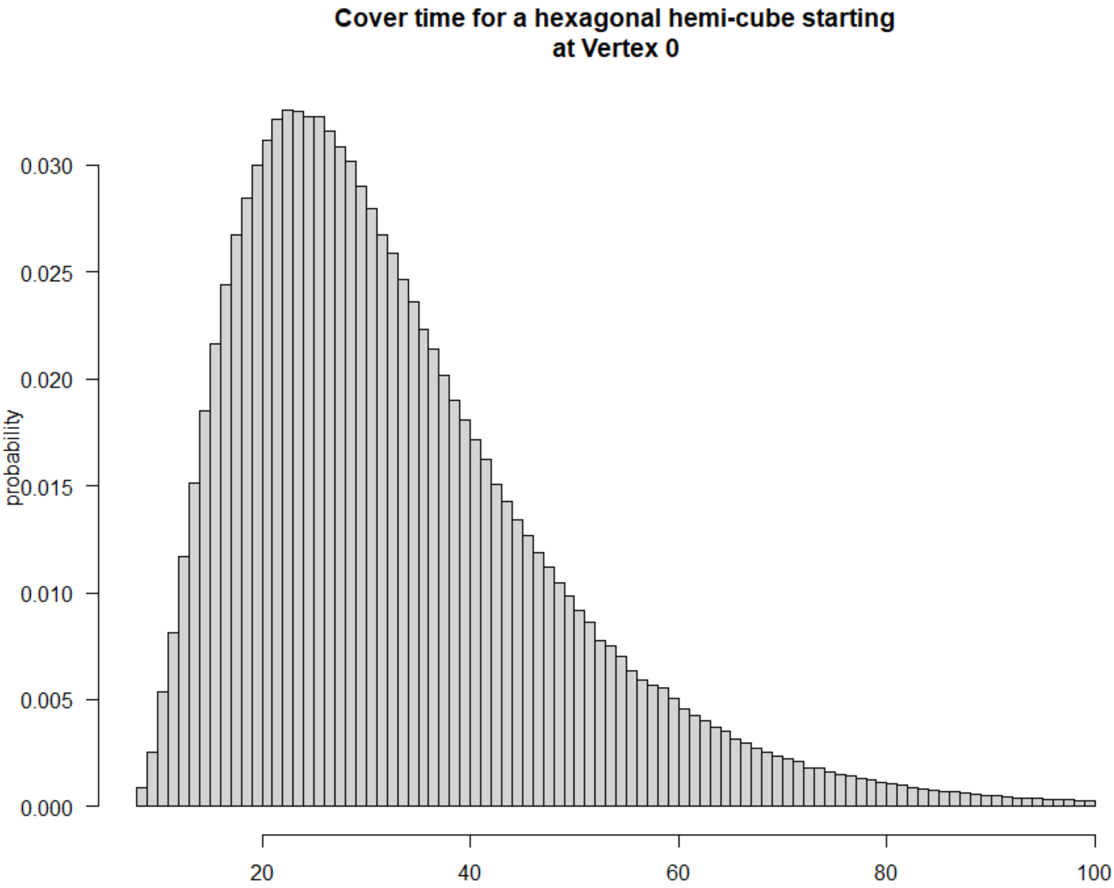

Figure 9: The simulated cover time distribution based on 10^6 iterations, truncated at 100 with $P(\bar{T} > 100) = .003082$. Some estimated quantiles are $q(.25) = 22, q(.50) = 30, q(.75) = 41, q(.99) = 85$.

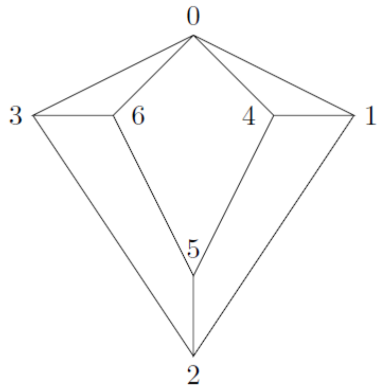

Figure 10: A planar representation of the rhombic hemi-cube obtained by a plane cut passing through the midpoints of a pair of opposite edges and containing either diagonal of the rectangle formed by the four vertices not on these edges.

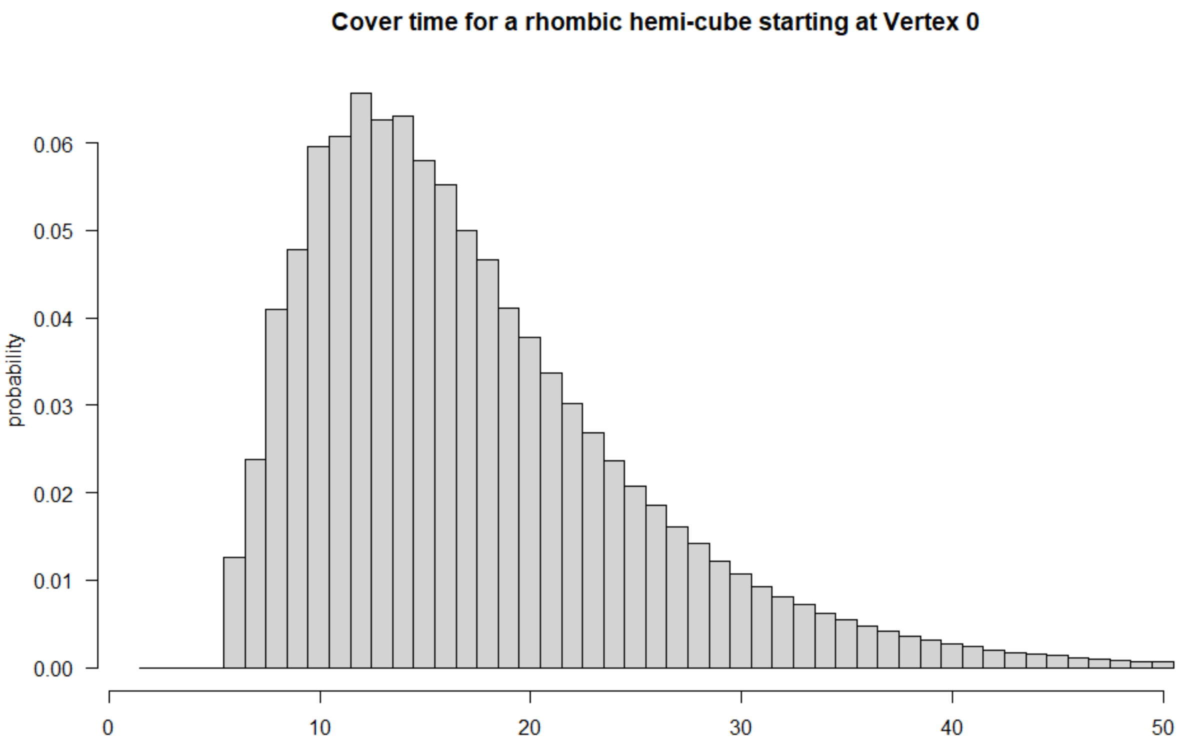

Figure 11: The simulated cover time distribution based on 10^6 iterations, truncated at 50 with $P(\bar{T} > 50) = .003959$. Some estimated quantiles are $q(.25) = 12, q(.50) = 16, q(.75) = 21, q(.99) = 44$.